Amplitude and Phase: Second Order II

The model. The Mathlet Amplitude and Phase: Second order II illustrates driving a spring/mass/dashpot system through the dashpot.

Suppose that the position of the bottom of the dashpot is given by $y(t)$, and that the mass is at $x(t)$, arranged so that $x = 0$ when the spring is relaxed. Then the force on the mass is given by

$$m\ddot{x} = -kx + b\frac{d}{dt}(y - x)$$

since the force exerted by a dashpot is supposed to be proportional to the speed of the piston moving through it. This can be rewritten

(1) $$m\ddot{x} + b\dot{x} + kx = b\dot{y}\,.$$
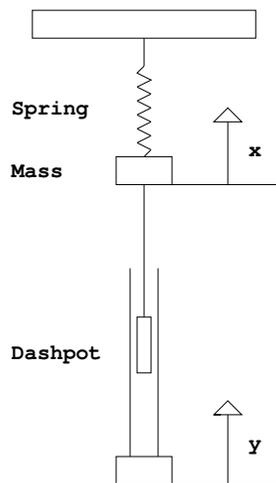
We will consider x as the system response, and the position of the back end of the dashpot, y, as the input signal. Note that the *derivative* of the input signal (multiplied by b) occurs on the right hand side of the equation. (Another system leading to the same mathematics is the series RLC circuit shown in the Mathlet `Series RLC Circuit`, in which the impressed voltage is the input variable and the voltage drop across the resistor is the system response.)

The solution. We suppose that the input signal is sinusoidal:

$$y = B\cos(\omega t)\,.$$

Then $\dot{y} = -\omega B\sin(\omega t)$ so our equation is

(2) $$m\ddot{x} + b\dot{x} + kx = -b\omega B\sin(\omega t)\,.$$

The Mathlet illustrates the case $B = 1$ and $m = 1$, but we will analyze the general case.

The sinusoidal system response will be of the form

$$x_p = gB\cos(\omega t - \phi)$$

for some gain g and phase lag ϕ, which we now determine by making a complex replacement.

The right hand side of (2) involves the sine function, so one natural choice would be to regard it as the imaginary part of a complex equation. This would work, but we should also keep in mind that the input signal is $B\cos(\omega t)$. For that reason, we will write (2) as the *real* part of a complex equation, using the identity $\operatorname{Re}(ie^{i\omega t}) = -\sin(\omega t)$. The equation (2) is thus the real part of

$$(3) \qquad\qquad m\ddot{z} + b\dot{z} + kz = bi\omega Be^{i\omega t}\ .$$

and the complex input signal is $Be^{i\omega t}$ (since this has real part $B\cos(\omega t)$).

The sinusoidal system response x_p of (2) is the real part of the exponential system response z_p of (3). The Exponential Response Formula gives

$$z_p = \frac{bi\omega}{p(i\omega)}Be^{i\omega t}$$

where

$$p(s) = ms^2 + bs + k$$

is the characteristic polynomial.

The complex gain is the complex number $W(i\omega)$ by which you have to multiply the complex input signal to get the exponential system response. Comparing z_p with $Be^{i\omega t}$, we see that

$$W(i\omega) = \frac{bi\omega}{p(i\omega)}\ .$$

As usual, write

$$W(i\omega) = ge^{-i\phi}$$

so that

$$z_p = W(i\omega)Be^{i\omega t} = gBe^{i(\omega t - \phi)}$$

Thus

$$x_p = \operatorname{Re}(z_p) = gB\cos(\omega t - \phi)$$

—the amplitude of the sinusoidal system response is g times that of the input signal, and lags behind the input signal by ϕ radians.

To make this more explicit, let's use the natural frequency $\omega_n = \sqrt{k/m}$. Then

$$p(i\omega) = m(i\omega)^2 + bi\omega + m\omega_n^2 = m(\omega_n^2 - \omega^2) + bi\omega \,,$$

so

$$W(i\omega) = \frac{bi\omega}{m(\omega_n^2 - \omega^2) + bi\omega}\,.$$

Thus the gain $g(\omega) = |W(i\omega)|$ and the phase lag $\phi = -\mathrm{Arg}(W(i\omega))$ are determined as the polar coordinates of the complex function of ω given by $W(i\omega)$. As ω varies, $W(i\omega)$ traces out a curve in the complex plane, shown by invoking the [Nyquist plot] in the applet.

Questions. 1. The Nyquist plot (for $\omega \geq 0$) appears to be independent of the values of the system parameters k and b (as was the case with the Nyquist plot of the system illustrated in `Amplitude and Phase: First Order`) and to form a circle. Is this really true? Why? If so, what is the center and radius of the circle?

2. Notice, from looking at the Bode plots, that the resonant peak seems to occur exactly when the phase lag is zero. Is this true? (The answer should be obvious from contemplating the Nyquist plot.)

3. The value of ω at which $\phi = 0$ (which, if the answer to **2.** is affirmative, is the resonant frequency ω_r) seems, from the Mathlet, to be independent of b. Is that the case? What is ω when $\phi = 0$, in terms of the natural frequency ω_n?

4. Fix k at some positive value, and push the resistance b towards 0. What do you observe about the Bode plot? For small b, what is the effect in the Nyquist plot as you sweep the frequency across the resonant frequency? How is this effect reflected in the Bode plots?

Let's be more quantitative. Notice that

$$|W(i\omega)|^2 = \frac{b^2\omega^2}{m^2(\omega_n^2 - \omega^2)^2 + b^2\omega^2} = \left(1 + \left(\frac{m}{b}\right)^2 \left(1 - \left(\frac{\omega_n}{\omega}\right)^2\right)^2\right)^{-1}$$

Suppose $b/m < 1$, and compute the *half-power points*, the values $\omega_{1/2}$ of ω for which

$$W(i\omega_{1/2}) = \frac{1}{2}$$

Using the tangent line approximation

$$(1+x)^{-1/2} \simeq 1 - \frac{x}{2}$$

estimate $|\omega_n - \omega_{1/2}|$ for b small.

5. For all values of k and b, as ω gets large it appears that $A \to 0$. Is this right? Can you be more precise about how? That is, for ω very large, can you approximate A by a simpler expression that what we derived above?—maybe just a (negative) power of ω?

6. For all values of k and b, as ω gets large it appears that $\phi \to \pi/2$. Is this right?

7. How about the behavior of A and ϕ for small values of ω? Clearly $g(0) = 0$ and $\phi(0) = -\pi/2$, so linear approximation gives

$$g(\omega) \simeq g'(0)\omega \quad , \quad \phi(\omega) \simeq -\frac{\pi}{2} + \phi'(0)\omega$$

for small ω. The Mathlet gives some indication of the values of $g'(0)$ and $\phi'(0)$. What are they in fact? Does the Mathlet bear out your calculation?

8. The Mathlet seems to indicate another similarity between this system and the first orders system studied in `Amplitude and Phase: First Order`: The maxima and minima of the system response occur just when the value of the system response equals the value of the input signal. Geometrically, the yellow and cyan curves cross exactly when the yellow curve has a horizontal tangent. Is this actually the case? Can you give a non-computational exponation?